2G-1 For each of these functions, on the indicated interval, find explicitly the point c whose existence is predicted by the Mean-value Theorem; if there is more than one such c, find all of them. Use the form (1).

(a) x^2 on $[0, 1]$ (~~b~~) $\ln x$ on $[1, 2]$ (c) $x^3 - x$ on $[-2, 2]$

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(b) $f(x) = \ln x$, $[1, 2]$

$$f'(x) = \frac{1}{x}$$

MVT: $$\frac{f(b) - f(a)}{b - a} = f'(c)$$

$$\Rightarrow \frac{f(2) - f(1)}{2 - 1} = \frac{1}{c}$$

$$\frac{\ln 2 - \ln 1}{1} = \frac{1}{c}$$

$$c = \frac{1}{\ln 2 - 0}$$
$$= \frac{1}{\ln 2}$$

2G-2 Using the form (2), show that

(a) $\sin x < x$, if $x > 0$ (~~b~~) $\sqrt{1+x} < 1 + x/2$ if $x > 0$.

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(b) $\sqrt{1+x} < 1 + \frac{x}{2}$, $x > 0$

$f(x) = \sqrt{1+x} - \left(1 + \frac{x}{2}\right)$

At $x = 0$, $f(0) = \sqrt{1+0} - (1+0) = 0$

$f'(x) = \frac{1}{2}\frac{1}{\sqrt{1+x}}(1) - \frac{1}{2}$

$= \frac{1}{2}\left(\frac{1}{\sqrt{1+x}} - 1\right)$

For $x > 0$, $\frac{1}{\sqrt{1+x}} < 1$.

$\Rightarrow f'(x) < 0$ for $x > 0$

$\Rightarrow f$ is decreasing for $x > 0$

$\Rightarrow f(x) < f(0)$

$\therefore \sqrt{1+x} - \left(1 + \frac{x}{2}\right) < 0 \iff \sqrt{1+x} < 1 + \frac{x}{2}$

for $x > 0$. $\square$

2G-5 a) Suppose $f''(x)$ exists on an interval I and $f(x)$ has a zero at three distinct points $a < b < c$ on I. Show there is a point p on $[a, c]$ where $f''(p) = 0$.

b) Illustrate part (a) on the cubic $f(x) = (x - a)(x - b)(x - c)$.

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a) $f(a) = 0$, $f(b) = 0$, $f(c) = 0$.

If $a < b < c$, then according to MVT,

$$\frac{f(b) - f(a)}{b - a} = f'(q_1) \text{ for some } q_1,\ a < q_1 < b.$$

$f(b) = f(a) = 0 \Rightarrow f'(q_1) = 0$

Similarly, $\dfrac{f(c) - f(b)}{c - b} = f'(q_2)$ for some q_2, $b < q_2 < c$.

And $f'(q_2) = 0$

Let $g = f'$, so $f'(q_1) = g(q_1)$ and $f'(q_2) = g(q_2)$.

Since $q_1 < b < q_2$, then $q_1 < q_2$.

$$\Rightarrow \frac{g(q_2) - g(q_1)}{q_2 - q_1} = g'(p) \text{ for some } p,\ q_1 < p < q_2.$$

$g(q_2) = g(q_1) = 0 \Rightarrow g'(p) = 0$

$g'(p) = f''(p)$. And since $a < q_1 < p < q_2 < c$,

$\therefore$ There is a point $p = [a, c]$, where $f''(p) = 0$.

$\square$

b) $f(x) = (x-a)(x-b)(x-c)$

$= (x^2 - ax - bx + ab)(x-c)$

$f'(x) = (2x - a - b)(x-c) + (x^2 - ax - bx + ab)(1)$

$= 2x^2 - ax - bx - 2cx + ac + bc$

$+ x^2 - ax - bx + ab$

$= 3x^2 - 2ax - 2bx - 2cx + ab + bc + ac$

$f''(x) = 6x - 2(a+b+c)$

$f'(q_1) = 3q_1^2 - 2(a+b+c)q_1 + ab + bc + ac$

$f'(q_2) = 3q_2^2 \cdots$

$f''(p) = 6p - 2(a+b+c)$

$\Rightarrow 6p = 2(a+b+c)$

$\therefore p = \frac{1}{3}(a+b+c)$

2G-6 Using the form (2) of the Mean-value Theorem, prove that on an interval $[a, b]$,

a) $f'(x) > 0 \quad \Rightarrow \quad f(x)$ increasing; (b) $f'(x) = 0 \quad \Rightarrow \quad f(x)$ constant.

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a) $f(b) = f(a) + f'(c)(b-a)$ for some c where $a < c < b$.

Let $a \leq x_1 < x_2 \leq b$.

$\Rightarrow f(x_2) = f(x_1) + f'(x)(x_2 - x_1)$ for some $x = [x_1, x_2]$.

$f(x_2) - f(x_1) = f'(x)(x_2 - x_1)$

$f'(x) > 0$ for all $a \leq x \leq b$.

$\Rightarrow f(x_2) - f(x_1) > 0$ for all $a \leq x_1 < x_2 \leq b$

since $x_2 - x_1 > 0$.

$\therefore f(x)$ is increasing.

(b) $f'(x) = 0$

$\Rightarrow f(x_2) - f(x_1) = 0$ for all $a \leq x_1 < x_2 \leq b$.

$\Rightarrow f(x_2) = f(x_1)$ in the interval $[a, b]$.

$\therefore f(x)$ is constant.

3A-1 Compute the differentials $df(x)$ of the following functions.

a) $d(x^7 + \sin 1)$ b) $d\sqrt{x}$ c) $d(x^{10} - 8x + 6)$

~~d)~~ $d(e^{3x} \sin x)$ ~~e)~~ Express dy in terms of x and dx if $\sqrt{x} + \sqrt{y} = 1$

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d) $d(e^{3x} \sin x)$

$= (3e^{3x} \sin x + e^{3x} \cos x)\, dx$

e) $\sqrt{x} + \sqrt{y} = 1$

$\sqrt{y} = 1 - \sqrt{x}$

$\Rightarrow d\sqrt{y} = \frac{1}{2\sqrt{y}} dy$, $d\sqrt{y} = -\frac{1}{2} \frac{1}{\sqrt{x}} dx$

$\Rightarrow dy = 2\sqrt{y}\, d\sqrt{y}$

$= 2(1-\sqrt{x})\left(-\frac{1}{2\sqrt{x}}\right)dx$

$= \left(\frac{-1}{\sqrt{x}} + 1\right)dx$

3A-2 Compute the following indefinite integrals

~~a)~~ $\int (2x^4 + 3x^2 + x + 8)dx$	b) $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx$	~~c)~~ $\int \sqrt{8+9x}dx$
d) $\int x^3(1-12x^4)^{1/8}dx$	~~e)~~ $\int \frac{x}{\sqrt{8-2x^2}}dx$	f) $\int e^{7x}dx$
~~g)~~ $\int 7x^4 e^{x^5}dx$	h) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$	~~i)~~ $\int \frac{dx}{3x+2}$
j) $\int \frac{x+5}{x}dx.$	~~k)~~ $\int \frac{x}{x+5}dx$. (Write $\frac{x}{x+5} = 1 + \ldots$.)	
l) $\int \frac{\ln x}{x}dx$	m) $\int \frac{dx}{x \ln x}$	

a) $\int (2x^4+3x^2+x+8)\,dx$

$= \frac{2x^5}{5} + \frac{3x^3}{3} + \frac{x^2}{2} + 8x + C$

$= \frac{2}{5}x^5 + x^3 + \frac{x^2}{2} + 8x + C$

c) $\int \sqrt{8+9x}\,dx$ Let $u = 8+9x$.
$\Rightarrow du = 9\,dx$

$= \int \frac{1}{9}\sqrt{u}\,du$

$= \frac{1}{9}\frac{u^{3/2}}{3/2} + C$

$= \frac{2u^{3/2}}{27} + C$

$= \frac{2(8+9x)^{\frac{3}{2}}}{27} + C$

e) Let $u = 8-2x^2$.
$\Rightarrow du = -4x\,dx$

$$\int \frac{x}{\sqrt{8-2x^2}}\,dx$$

$$= -\frac{1}{4}\int \frac{1}{\sqrt{u}}\,du$$

$$= -\frac{1}{4}\,\frac{\sqrt{u}}{1/2} + C$$

$$= -\frac{1}{2}\sqrt{8-2x^2} + C$$

g) Let $u = e^{x^5}$.
$du = e^{x^5}\cdot 5x^4\,dx$

$$\int 7x^4 e^{x^5}\,dx$$

$$= \int \frac{1}{5\cancel{x^4}}\,7\cancel{x^4}\,du$$

$$= \int \frac{7}{5}\,du$$

$$= \frac{7}{5}u + C$$

$$= \frac{7}{5}e^{x^5} + C$$

i) $$\int \frac{1}{3x+2}\,dx$$

$$= \ln|3x+2|\cdot\frac{1}{3} + C$$

$$= \frac{1}{3}\ln|3x+2| + C$$

k) $$\int \frac{x}{x+5}\,dx$$

$$\begin{array}{r} 1 \\ x+5\,\overline{)\,x+0} \\ \underline{x+5} \\ -5 \end{array}$$

$$= \int 1 + \frac{-5}{x+5}\,dx$$

$$= x - 5\ln|x+5| + C$$

3A-3 Compute the following indefinite integrals.

~~a)~~ $\int \sin(5x)dx$ b) $\int \sin(x)\cos(x)dx$ ~~c)~~ $\int \cos^2 x \sin x dx$

d) $\int \frac{\cos x}{\sin^3 x}dx$ ~~e)~~ $\int \sec^2(x/5)dx$ f) $\int \tan^6 x \sec^2 x dx$

~~g)~~ $\int \sec^9 x \tan x dx$

a) $\int \sin(5x)\,dx$

$= -\cos 5x \cdot \frac{1}{5} + C$

$= -\frac{\cos 5x}{5} + C$

c) $\int \cos^2 x \sin x \, dx$ $u = \cos x$
$du = -\sin x \, dx$

$= \int -u^2 \, du$

$= -\frac{u^3}{3} + C$

$= -\frac{\cos^3 x}{3} + C$

e) $\int \sec^2\left(\frac{x}{5}\right) dx$

$= 5 \tan \frac{x}{5} + C$

g) $\int \sec^9 x \tan x \, dx$

Let $u = \cos x$.
$du = -\sin x \, dx$

$= -\int \frac{1}{u^9} \frac{1}{u} \, du$

$= -\frac{1}{-9} \cdot \frac{1}{u^9} + C$

$= \frac{1}{9\cos^9 x} + C$

$= \frac{1}{9} \sec^9 x + C$

3F-1 Solve the following differential equations

a) $dy/dx = (2x+5)^4$ b) $dy/dx = (y+1)^{-1}$

~~c)~~ $dy/dx = 3/\sqrt{y}$ ~~d)~~ $dy/dx = xy^2$

c) $\frac{dy}{dx} = \frac{3}{\sqrt{y}}$

$\int \sqrt{y}\, dy = \int 3\, dx$

$\frac{y^{3/2}}{3/2} = 3x + c_1$

$y^{3/2} = \frac{9}{2}x + C$

$y = \left(\frac{9}{2}x + C\right)^{\frac{2}{3}}$

d) $\frac{dy}{dx} = xy^2$

$\int \frac{1}{y^2}\, dy = \int x\, dx$

$-\frac{1}{y} = \frac{x^2}{2} + c_1$

$y = -\frac{2}{x^2 + C}$

3F-2 Solve each differential equation with the given initial condition, and evaluate the solution at the given value of x:

a) $dy/dx = 4xy$, $y(1) = 3$. Find $y(3)$.

e) $dy/dx = e^y$, $y(3) = 0$. Find $y(0)$. For which values of x is the solution y defined?

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a) $\frac{dy}{dx} = 4xy$, $y(1) = 3$

$$\int \frac{1}{y}\,dy = \int 4x\,dx$$

$$\ln|y| = \frac{4x^2}{2} + C_1$$

$$|y| = e^{2x^2} \cdot e^{C_1}$$

$$y = \pm e^{C_1} \cdot e^{2x^2}$$

$$= Ce^{2x^2}$$

$y(1)=3 \Rightarrow 3 = Ce^2$

$$C = \frac{3}{e^2}$$

$\therefore$ $y = 3e^{2x^2-2}$

$y(3) = 3e^{16}$

e) $\frac{dy}{dx} = e^y$, $y(3) = 0$

$$\int \frac{dy}{e^y} = \int dx$$

$$\int e^{-y}\,dy = \int dx$$

$$\frac{e^{-y}}{-1} = x + C$$

$$e^{-y} = -x - C$$

$$-y = \ln(-x-C)$$

$$y = -\ln(-x-C)$$

$\Rightarrow 0 = -\ln(-3-C)$

$$-3-C = e^0$$

$$C = -3-1$$

$$= -4$$

$\therefore$ $y = -\ln(-x+4)$

y is defined for $x > 4$

3F-4 Newton's law of cooling says that the rate of change of temperature is proportional to the temperature difference. In symbols, if a body is at a temperature T at time t and the surrounding region is at a constant temperature T_e (e for external), then the rate of change of T is given by

$$dT/dt = k(T_e - T).$$

The constant $k > 0$ is a constant of proportionality that depends properties of the body like specific heat and surface area.

a) Why is $k > 0$ the only physically realistic choice?

~~b)~~ Find the formula for T if the initial temperature at time $t = 0$ is T_0.

~~c)~~ Show that $T \to T_e$ as $t \to \infty$.

~~d)~~ Suppose that an ingot leaves the forge at a temperature of 680° Celsius in a room at 40° Celsius. It cools to 200° in eight hours. How many hours does it take to cool from 680° to 50°? (It is simplest to keep track of the temperature difference $T - T_e$, rather than T. The temperature difference undergoes exponential decay.)

e) Suppose that an ingot at 1000° cools to 800° in one hour and to 700° in two hours. Find the temperature of the surrounding air.

f) Show that $y(t) = T(t - t_0)$ also satisfies Newton's law of cooling for any constant t_0. Write out the formula for $T(t - t_0)$ and show that it is the same as the formula in E10/17 for $y(t)$ by identifying the constants k, T_e and T_0 with their corresponding values in the displayed formula in E10/17.

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b) $\dfrac{dT}{dt} = k(T_e - T)$, $T(0) = T_0$

$$\int \frac{dT}{T_e - T} = \int k\,dt$$

$$\frac{\ln|T_e - T|}{-1} = kt + C_1$$

$$T_e - T = \pm e^{-(kt + C_1)} = Ce^{-kt}$$

$$T = T_e - Ce^{-kt}$$

$$T_0 = T_e - Ce^{-0}$$

$$C = T_e - T_0$$

$$\rightarrow \quad \therefore T = T_e - (T_e - T_0)e^{-kt}$$

$$= T_e - T_e e^{-kt} + T_0 e^{-kt}$$

$$= T_e(1 - e^{-kt}) + T_0 e^{-kt}$$

c) As $t \to \infty$,

$$T = T_e(1-0) + T_0(0)$$
$$= T_e$$

d) $T_e = 40$, $T_0 = 680$

$$T = 40(1-e^{-kt}) + 680e^{-kt}$$

$T = 200$, $t = 8$

$$\Rightarrow 200 = 40 - 40e^{-8k} + 680e^{-8k}$$
$$640e^{-8k} = 200 - 40$$
$$e^{-8k} = \frac{160}{640} = \frac{1}{4}$$
$$-8k = \ln\left(\frac{1}{4}\right)$$
$$k = \frac{\ln 4}{8}$$

$T = 50$

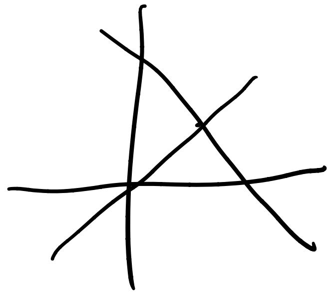
$$\Rightarrow 50 = 40 - 40e^{-\frac{\ln 4}{8}t} + 680e^{-\frac{\ln 4}{8}t}$$
$$640e^{-\frac{\ln 4}{8}t} = 10$$
$$-\frac{\ln 4}{8}t = \ln\left(\frac{1}{64}\right)$$
$$t = \ln 64 \times \frac{8}{\ln 4}$$
$$= 8\frac{\ln 64}{\ln 4}$$
$$= 8\frac{\ln 4^2 + \ln 4}{\ln 4}$$
$$= 8(2+1)$$
$$= 24$$

3F-8 a) Find all plane curves such that the tangent line at P intersects the x-axis 1 unit to the left of the projection of P on the the x-axis.

~~b)~~ Find all plane curves in the first quadrant such that for every point P on the curve, P bisects the part of the tangent line at P that lies in the first quadrant.

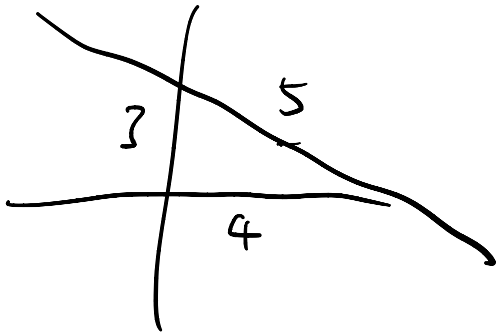
b)

$$y = \frac{dy}{dx} x + C$$



$$0 = \frac{dy}{dx} x + C$$

$$\frac{dy}{dx} = -\frac{C}{x}$$



$$2^2 + \left(\frac{3}{2}\right)^2$$

$$4 + \frac{9}{4}$$

$$\frac{16}{4} \quad \frac{25}{2}$$